

## Introduction and Fundamentals for Mathematics and Continuum Mechanics

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### 1.1 General Introduction

In this chapter, the fundamental properties that make asphalt concrete (AC) different from other materials are discussed. Logically, this discussion will serve as the guiding principles for the entire book.

Asphalt concrete is a unique material in four aspects: 1) its composition; 2) its manufacturing process; 3) its use environment; and 4) its performance or failure modes. These unique characteristics make AC different from other materials, and make this book valuable and uncommon. There are many textbooks on the mechanics of different materials and many books on the general topics of mechanics as well. What makes AC different from other materials is addressed in detail and serves as the focus of this book.

#### 1.1.1 Unique Material Structure and Complexity

AC consists mainly of three constituents: asphalt binder, aggregates, and air voids. Other constituent/constituents may include additives such as fillers, modifiers, and fibers. These constituents are often used to enhance the properties of AC for specific aspects, such as moisture damage. The three major constituents have great differences in their properties and variety. While most of the binders are residuals of petroleum refinery, different petroleum resources and refining procedures will end up with binders of different molecular structure and compositions. Aggregates come with a variety of minerals that have different reactions with binders. The stiffness of the three constituents is so drastically different that tremendous localizations make properties of this material highly variable. While there are other distinct properties, three important properties of binder—thermal sensitivity, loading rate sensitivity, and moisture sensitivity—make this material much more challenging than most other materials in characterization, modeling and simulation, and especially predicting its behavior or performance under complex use environments including temperature, loading magnitude, and loading rates. The unique material structure and complexity requires a microscopic view.

In addition, due to the significant differences in chemical composition between binder and aggregates, the bonding between asphalt binder and aggregates is weak and has not been well understood. So far, there are not enough experimental and theoretical backgrounds pinpointing the behavior of interfaces between aggregates and binder, nor between fillers and binder.

### **1.1.2 Construction**

Unlike other materials such as polymer and metals, the production of AC for real roads presents difficulties in controlling its quality in the areas of: 1) aggregate moisture; 2) the fines stuck at the aggregate surfaces; 3) temperature; 4) wind and moisture conditions at compaction; and 5) roadbed conditions. The variations in these conditions affect the bonding between aggregates and binder, binder content, void distribution, and even segregation. More importantly, a typical lift thickness is around three times the nominal maximum size of the aggregates, smaller than the required size for a representative volume, making compaction difficult and uniformity hard to achieve. These factors significantly limit understanding or predicting the behavior of asphalt pavement.

### **1.1.3 Environment Exposure**

Roads must be in a natural environment without protection or isolation. Variations in temperature, moisture conditions, sunshine exposure, and the loading spectrum (loading magnitude, speed, and interval) are difficult to characterize, making predicting the performance of AC much more challenging than most other engineering materials. In addition, the “strength” of AC is usually degrading (due to aging, sometimes also increasing due to healing) with time, and since the degrading mechanisms are very complicated to explore, they are not well understood.

### **1.1.4 Failure Modes**

Unlike many other engineering structures, roads are not designed against failures such as fracture or limited elastic deformation that can be accurately determined. They are designed against several distresses such as rutting (permanent deformation), fatigue cracking, thermal cracking, and roughness. These distresses accumulate with the repetitions of loads and thermal cycles. Due to the complexities described in previous sections, the performance characterized in the laboratory can hardly be used to predict road performance. This makes empirical methods, such as the mechanistic-empirical (M-E) method, necessary.

### **1.1.5 Moisture Damage**

A special distress of AC is moisture damage. Moisture entrapped in AC produces large excess pore water pressure. When asphalt film is soaked in water, its light components may be dissolved and become weak. Moisture infused between aggregates and binder interface will also weaken this interface. Moisture damage causes the removal of fine materials, stripping the binder from the aggregate surface, and thus changing the material structure. This phenomenon is not well understood.

### **1.1.6 Complex Coupling**

Experiments have also discovered that rutting induces fatigue cracking and vice versa. Moisture damage, weakening the mastics and binder-aggregate interface, leads to acceleration of fatigue and rutting.

The complexities described above can be summarized as the effect of non-linearity, non-uniformity, rate dependency, temperature dependency, large deformation, and complex coupling in terms of mechanics. The overall consequence is that the prediction of the pavement performance based on properties characterized in laboratories may be several magnitudes different from the performance observed in the field.

While the author began with an objective to document most of the studies relevant to the mechanics of AC, the study identified thousands of articles and reports. It is not the intent of this book to document all these studies, which may require an independent book of several hundred pages. Instead, this book will focus on the more rigorous theories of mechanics and their related applications. Laboratory discoveries, empirical correlation, relatively simple models such as isotropic elasticity models, and viscoelasticity models are not discussed in this book. Since systematic and fundamental research looking into the mechanisms of deformation and failure, stress-path dependency, and anisotropy is not available, it is difficult to present the book in such a way where concepts in different chapters are tied together to develop the various distress models.

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## 1.2 Phenomenological Behavior of Asphalt

Phenomenologically, AC demonstrates behavior of non-linearity, rate dependency, temperature sensitivity, anisotropy, heterogeneity, irrecoverable deformation, fracturing, healing, hardening, softening, dilatancy, etc. Theories to cover all these phenomena and their couplings are too complicated to be realistic. Phenomenological models that describe AC behavior are typically elasticity, anisotropic elasticity, non-linear elasticity, elastoplasticity, linear or non-linear viscoelasticity, viscoplasticity, fracture mechanics, and continuum damage mechanics.

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## 1.3 Need for This Book

The need for a book such as this one is two-fold: industrial and educational. Historically, the characterization of AC has been dominated by empirical approaches. In the last two decades an (M-E) approach for pavement design has been developed. Under this driving force, more rigorous theories have been developed or applied to solve problems encountered in asphalt industries. The variety of theories is so large that even researchers in this field would become lost seeking the links among these theories and understanding their capabilities and limits. An introduction to the fundamentals of these theories would be helpful to industrial researchers and practicing engineers to identify viable methods. The educational needs are relatively straightforward, as was indicated in the preface. So far, there is no such textbook. It is anticipated the book could serve as a textbook for characterization, modeling, and simulation courses on AC and pavements.

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## 1.4 Logical Link of Chapters

There are many books in mechanics dealing with topics such as stress, strains, kinematics, tensor analysis, and fundamental laws of conservation and objectivity. Excellent books are available on these topics. There are also excellent textbooks on elasticity, viscoelasticity, plasticity, viscoplasticity, fracture mechanics, continuum damage mechanics

molecular dynamics, and numerical implementations in Finite Element Method (FEM), Discrete Element Method (DEM) and Boundary Element Method (BEM). In the interest of being concise, only a brief description of these fundamentals is included in Chapter 1, Chapter 5, Chapter 6, Chapter 8, Chapter 9 and Chapter 13 for completeness. Readers are assumed to have a background in elasticity and tensor analysis and the required mathematics background (such as Laplace Transform) to read these textbooks if needed. With this understanding, the book is organized into six sections with a total of 13 chapters with both straightforward fundamentals and applications so that practicing engineers and graduate students can learn this topic with convenience.

The first section has two chapters (Chapters 1 and 2) to discuss the fundamental properties of AC, fundamentals of tensor analysis and continuum mechanics, and the constituent properties and characterization methods. The second section has three chapters (Chapters 3, 4, and 5) focusing on microstructure characterization, strain characterization at the microscopic level, and mixture theory and micromechanics applications. The third section has two chapters (Chapters 6 and 7) to discuss macro-continuum types of theories and viscoplasticity models developed for AC. The fourth section has two chapters (Chapters 8 and 9) to mainly focus on three major numerical methods including FEM, BEM, and DEM. Section five has three chapters (Chapters 10, 11, and 12) to discuss topics of active research areas with applications of the fundamental theories and techniques. Section six has one chapter (Chapter 13) on multi-scale modeling and topics of current interest and less matured. Appendices document the Laplace Transform, Eshelby Tensors for elliptic inclusions, and fundamental solutions for BEM.

A list of the books recommended for further reading is presented at the end of this chapter. Again, the philosophy is not to perform a literature review or evaluation of these books. Instead, the goal is to assist the readers of this book to pick up the backgrounds quickly, so only a few books in each category are listed. One of the most important selection criteria for these books was simplicity, conciseness, and description of the fundamental physical mechanisms.

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## 1.5 Fundamentals of Mathematics

### 1.5.1 Scalar

A scalar represents physical or geometrical quantities that require only a magnitude to represent. Quantities such as time, temperature, density, and distance can be represented by scalars. It is also called zeroth-order-tensor, requiring  $3^0 = 1$  basis (independent direction). In this chapter, lower-case Greek letters such as  $\alpha$ ,  $\beta$  and  $\gamma$  are used to represent scalars. In fact, one may judge whether a quantity in an equation is a scalar, a vector, or a tensor from its physical meaning and dimensional analysis. The algebra operations such as + and – can be applied to scalars.

Geometrically, scalars are one-dimensional (1D) quantities that have zero as a reference point and positive and negative “directions” along only one axis.

### 1.5.2 Vector

Physical quantities such as force and displacement require both a magnitude and a direction to represent. These quantities can be represented as vectors, generally having three independent bases to represent in the three-dimensional (3D) space. If one

compresses one or two of the three dimensions to zero, one can have two-dimensional (2D) vectors and a 1D vector (a scalar). A vector is also called a first-order tensor, having  $3^1 = 3$  independent bases. Mathematically, a set of mutually independent  $N$  variables can be considered as an  $N$  dimensional vector. Lower-case characters such as  $v$  and  $u$  are used to represent vectors in this chapter. Vectors can be also represented in indicial format as  $v_i$  and  $u_i$  ( $i = 1\sim 3$ ). Vectors in rigorous sense (2D and 3D) follow the triangle rule for “add” and “subtract.” Abstract extension to an  $N$  dimensional vector may not follow such rules.

A vector can be represented in its indicial format as follows:

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 = \sum_{i=1}^3 v_i e_i \tag{1-1}$$

Where  $e_i$  are the three bases (with unit length). By following the Einstein convention (the dummy index or the repeated index will sum running from 1 to 3), it can be simply represented as Equation 1-2:

$$v = v_i e_i \tag{1-2}$$

$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$  is the magnitude of a vector or the “normal/mod/length” of a vector (see Figure 1.1a). The vector can be generally represented as scalar (magnitude) in an orientation where  $e$  is the orientation. It is this orientation that needs a reference system to measure it. The above representation (Equation 1-1) is for a Cartesian reference system.

Addition or subtraction of vectors will follow the following rules (Figure 1.1b):

$$w = u \pm v, w_i e_i = (u_i \pm v_i) e_i \tag{1-3}$$

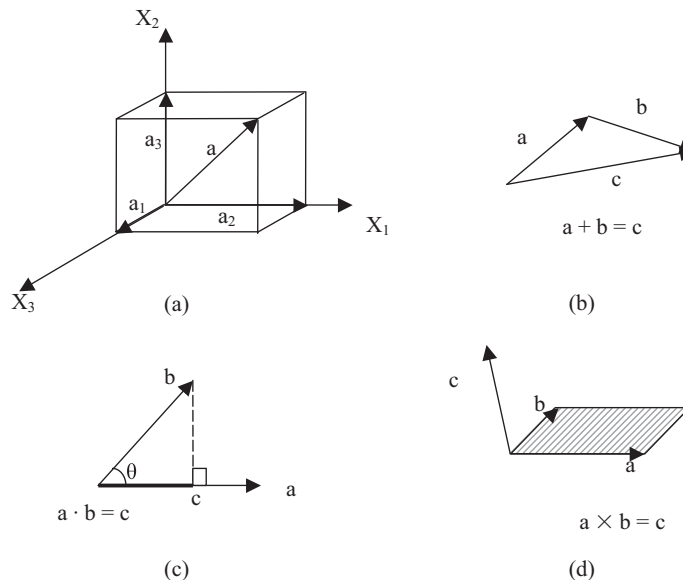


FIGURE 1.1 Illustration of typical vector operations.

Multiplication of a vector by a scalar is equivalent to extension or contraction of the vector along its original orientation (may reverse its direction, or make it a zero vector, which may not have a physical meaning).

$$\lambda v = \lambda v_i e_i \quad (1-4)$$

The dot (scalar) product of two vectors results in a scalar (the magnitude of the projection of one vector on the other) (Figure 1.1c).

$$u \bullet v = v \bullet u = \|u\| \|v\| \cos \theta = u_i v_i \quad (1-5)$$

Where  $\theta$  is the angle between the two vectors.

The cross product of two vectors is a vector with its magnitude equal to  $\|u\| \|v\| \sin \theta$  and its orientation normal to the plane formed by the two vectors and following the right-hand convention (Figure 1.1d). The magnitude is equal to the area of the parallelograms as shown in Figure 1.1d.

$$u \times v = -v \times u = \|u\| \|v\| \sin \theta |e| \quad (1-6)$$

The indicial format of the cross-product will be presented in the next section as it requires the use of some tensors.

### 1.5.3 Tensor

Many quantities in mechanics require more than three independent components to represent. The stresses and strains are good examples. They can be represented as second-order tensor (rank of two), with a total of nine components ( $3^2 = 9$ ). The nine components may not necessarily be independent. In this chapter, capital characters such as  $T$ ,  $Q$ , and  $C$  are used to represent tensors. The indicial format of the tensors (second-order) may be represented as  $T_{ij}$  and  $Q_{ij}$ .

Two important tensors, the Kronecker Delta and the Permutation Tensor, are first defined as they are often used in other vector and tensor operations.

#### 1.5.3.1 Kronecker Delta

Kronecker Delta is equivalent to the unit (often represented as unit isotropic tensor  $I$ ) or one in scalar. It also serves as an operator (some tensors such as the transformation tensor also serve as operators).

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, i = 1-3; j = 1-3 \\ 0 & \text{for } i \neq j, i = 1-3; j = 1-3 \end{cases} \quad (1-7)$$

It can be conveniently verified that:

$$e_i \bullet e_j = \delta_{ij} \quad (i, j = 1, 2, 3)$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

$$\delta_{ij} e_j = e_i$$

The last property represents the replacement operation.

Using the Kronecker Delta, the scalar product can be represented as:

$$u \bullet v = v \bullet u = u_i v_j e_i \bullet e_j = u_i v_j \delta_{ij} = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$$

### 1.5.3.2 Permutation Tensor

The permutation tensor,  $\xi_{ijk}$ , is a rank three tensor. It has  $3^3 = 27$  components as defined in the following:

$$\xi_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ appears as in the sequence } 123123 \\ -1 & \text{if } ijk \text{ appears as in the sequence } 321321 \\ 0 & \text{if } ijk \text{ appears as in any other sequences} \end{cases} \quad (1-8)$$

One can conveniently verify that:  $\xi_{miq} \xi_{jkq} = \delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ij}$  and  $\xi_{ijk} = -\xi_{jik}$

It can also be conveniently verified that the cross product of the bases can be represented as:

$$e_i \times e_j = \xi_{ijk} e_k \quad (i, j, k = 1, 2, 3)$$

Therefore, the cross product of two vectors can be expressed as:

$$u \times v = (u_i e_i) \times (v_j e_j) = u_i v_j (e_i \times e_j) = u_i v_j \xi_{ijk} e_k$$

and  $u \times v = -v \times u$ , making use of  $\xi_{ijk} = -\xi_{jik}$ .

### 1.5.3.3 Dyadic Product of Two Vectors

The dyadic product of two vectors,  $a \otimes b$ , is defined as a linear operation that makes two vectors into a tensor. It has the following features ( $a$ ,  $b$ , and  $c$  are vectors;  $\alpha$  is a scalar).

$$(\alpha a) \otimes b = a \otimes (\alpha b) = \alpha (a \otimes b) \quad (1-9)$$

$$a \otimes (b + c) = a \otimes b + a \otimes c \quad (1-10)$$

$$(b + c) \otimes a = b \otimes a + c \otimes a \quad (1-11)$$

$$a \otimes b = (a_i e_i) \otimes (b_j e_j) = a_i b_j e_i \otimes e_j \quad (1-12)$$

In more rigorous mathematics, a tensor is defined as a linear combination of the basis dyads (Equation 1-13).

$$A = A_{ij} e_i \otimes e_j \quad (1-13)$$

### 1.5.3.4 Product of Dyad with Vector

The inner product of a dyad with a vector is defined as  $(a \otimes b) \bullet c = a(b \bullet c)$ . Its indicial format is as follows:

$$\begin{aligned} (a \otimes b) \bullet c &= a_i e_i (b_j e_j \bullet c_k e_k) = a_i e_i (b_j \bullet c_k \delta_{jk}) = a_i b_k c_k e_i = a_1 (b_1 c_1 + b_2 c_2 + b_3 c_3) e_1 + \\ & a_2 (b_1 c_1 + b_2 c_2 + b_3 c_3) e_2 + a_3 (b_1 c_1 + b_2 c_2 + b_3 c_3) e_3 \end{aligned} \quad (1-14)$$

Obviously, the inner product ends up with a vector. The physical meaning is the projection of a tensor on the surface to give the surface traction (see the stress-surface traction relationship).

$$a \bullet (b \otimes c) = (a_i e_i \bullet b_j e_j) c_k e_k = a_i b_j \delta_{ij} c_k e_k = a_i b_i c_k e_k = (a_1 b_1 + a_2 b_2 + a_3 b_3) c_1 e_1 + (a_1 b_1 + a_2 b_2 + a_3 b_3) c_2 e_2 + (a_1 b_1 + a_2 b_2 + a_3 b_3) c_3 e_3 \quad (1-15)$$

Other vector dyad products may include:

$$a \times (b \otimes c) = (a_i e_i \times b_j e_j) c_k e_k = \xi_{ijq} a_i b_j c_k e_q \otimes e_k \quad (1-16)$$

$$(a \otimes b) \times c = a_i e_i (b_j e_j \times c_k e_k) = \xi_{jki} a_i b_j c_k e_i \otimes e_q \quad (1-17)$$

### 1.5.3.5 Dyad-Dyad Product

The dyad-dyad product of two dyads is defined as:

$$(u \otimes v) \bullet (w \otimes s) = u_i e_i (v_j e_j \bullet w_k e_k) s_q e_q = u_i v_j w_k s_q e_i \otimes e_q \quad (1-18)$$

It is a dyad.

### 1.5.3.6 Vector-Tensor Products

$$v \bullet A = v_i e_i \bullet A_{jk} e_j \otimes e_k = v_i A_{jk} \delta_{ij} e_k = v_i A_{ik} e_k \quad (1-19)$$

$$A \bullet v = A_{ij} e_i \otimes e_j \bullet v_k e_k = A_{ij} e_i \delta_{jk} v_k = A_{ij} v_j e_i \quad (1-20)$$

If  $A$  is symmetric, it can be conveniently verified that the above two products are the same.

This product carries the meaning of the projection of a tensor (stress) to the surface as surface traction. In that case,  $A$  is the stress tensor and  $v$  is the direction normal of the surface.

### 1.5.3.7 Tensor-Tensor Product

$$A \bullet S = A_{ij} e_i \otimes e_j \bullet S_{pq} e_p \otimes e_q = A_{ij} S_{jq} e_i \otimes e_q \quad (1-21)$$

The product is a tensor. It follows the same rule as the dyad-dyad product in 1.5.3.5.

## 1.5.4 Frame Transformation

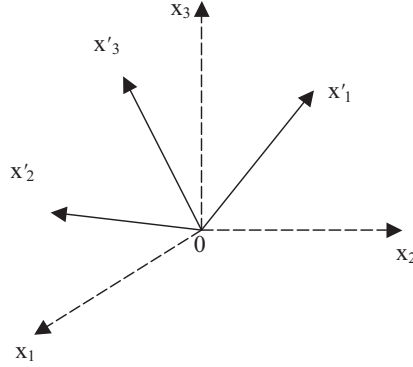
A scalar will not change its value when the frame is changed, or it is frame independent. However, vectors and tensors will follow certain rules when the frame is changed.

### 1.5.4.1 Vector Transformation

Considering two-frame systems  $(x_1, x_2, x_3)$  and  $(x'_1, x'_2, x'_3)$  (Figure 1.2), their basis vectors are corresponding  $(e_1, e_2, e_3)$  and  $(e'_1, e'_2, e'_3)$ . From algebra geometry, one can represent the relationship of the two basis systems, or the coordinates of the basis vectors in the original frame in the following equation:

$$\begin{Bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{Bmatrix} = \begin{Bmatrix} \cos(1'-1), \cos(1'-2), \cos(1'-3) \\ \cos(2'-1), \cos(2'-2), \cos(2'-3) \\ \cos(3'-1), \cos(3'-2), \cos(3'-3) \end{Bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix}$$

**FIGURE 1.2** Illustration of frame rotation ( $O x_1 x_2 x_3$  rotated to  $O x'_1 x'_2 x'_3$ ).



or

$$e'_i = T_{ij} e_j \quad e'_i = T_{ij} e_j \quad (1-22)$$

Considering the two basis vectors  $e'_i = T_{ir} e_r$  and  $e_j = T_{js} e_s$ , the scalar product of the two basis vectors is:

$$e'_i \cdot e'_j = T_{ir} e_r \cdot T_{js} e_s = T_{ir} T_{js} e_r \cdot e_s = T_{ir} T_{js} \delta_{rs} = T_{ir} T_{jr} = \delta'_{ij} = \delta'_{ij}$$

By the nature of the directional cosines, one can verify that  $T_{ir} T_{jr} = \delta'_{ij}$ .

It can be deduced that any vector  $v$  can be transformed into  $v'$  following frame transformation.

$$\begin{Bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{Bmatrix} = \begin{Bmatrix} \cos(1'-1), \cos(1'-2), \cos(1'-3) \\ \cos(2'-1), \cos(2'-2), \cos(2'-3) \\ \cos(3'-1), \cos(3'-2), \cos(3'-3) \end{Bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

or

$$v' = T v \quad v'_i = T_{ij} v_j \quad (1-23)$$

#### 1.5.4.2 Tensor Transformation for Different Coordinate Systems

Considering a tensor  $A = A_{ij} e_i \otimes e_j$  in  $(e_1, e_2, e_3)$  and its representation in the frame  $(e'_1, e'_2, e'_3)$ .

$$A'_{ij} e'_i \otimes e'_j$$

Clearly, one has  $e'_i = T_{ir} e_r$  and  $e'_j = T_{js} e_s$  and therefore:

$$A'_{ij} T_{ir} e_i \otimes T_{js} e_s = A'_{ij} T_{ir} T_{js} e_r \otimes e_s$$

Therefore,

$$A_{rs} = A'_{ij} T_{ir} T_{js} \quad (1-24)$$

#### Isotropic Tensor

A second order isotropic tensor can be represented as:

$$\begin{aligned} I &= \delta_{ij} e_i \otimes e_j \\ \delta_{ij} T_{ir} T_{js} e'_r \otimes e'_s &= T_{jr} T_{js} e'_r \otimes e'_s = \delta_{rs} e'_r \otimes e'_s \\ \delta'_{ij} &= \delta_{pq} T_{pi} T_{qj} = T_{qi} T_{qj} = \delta_{ij} \end{aligned} \quad (1-25)$$

Therefore, it does not vary with the frame changes. In analogy, it is equivalent to one of the scalars.

#### Fourth Order Isotropic Tensor

Any of the fourth order isotropic tensors can be represented as the combinations of the following three fourth order isotropic tensors:

$$\delta_{ij}\delta_{kl}e_i \otimes e_j \otimes e_k \otimes e_l$$

$$\delta_{ik}\delta_{jl}e_i \otimes e_j \otimes e_k \otimes e_l$$

$$\delta_{il}\delta_{jk}e_i \otimes e_j \otimes e_k \otimes e_l$$

or

$$(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk})e_i \otimes e_j \otimes e_k \otimes e_l \quad (1-26)$$

### 1.5.5 Matrix Operations

The second order tensors are often represented as  $3 \times 3$  matrices and all the matrix operations are valid for the tensors. Among many of the operations, the operation to find the principal directions and principal values of matrices has significances in determining the principal directions and values of stresses. Considering a tensor  $A$ , its projection onto the  $\mathbf{u}$  direction is a vector  $\mathbf{v}$ .

$$A_{ij}u_j = v_i \text{ or } A \bullet \mathbf{u} = \mathbf{v} \quad (1-27)$$

A special case is that the projection on to the  $\mathbf{n}$  direction ends up with a vector that is in line with  $\mathbf{n}$  direction.

$$A_{ij}n_j = \lambda n_i \text{ or } A \bullet \hat{\mathbf{n}} = \lambda \hat{\mathbf{n}} \quad (1-28)$$

or

$$(A_{ij} - \lambda\delta_{ij})n_j = 0 \text{ or } (A - \lambda I) \bullet \hat{\mathbf{n}} = 0 \quad (1-29)$$

Therefore, one has the following equations:

$$(A_{11} - \lambda)n_1 + A_{12}n_2 + A_{13}n_3 = 0$$

$$A_{21}n_1 + (A_{22} - \lambda)n_2 + A_{23}n_3 = 0$$

$$A_{31}n_1 + A_{32}n_2 + (A_{33} - \lambda)n_3 = 0 \quad (1-30)$$

The solution to  $\lambda$  is equivalent to the following matrix equation:

$$\left| A_{ij} - \lambda\delta_{ij} \right| = 0 \quad (1-31)$$

or

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \quad (1-32)$$

Where  $I_1$  (first invariant, the trace),  $I_2$  (second invariant), and  $I_3$  (third invariant) are the invariants of the tensor (matrix).

$$I_1 = A_{ii} = \text{tr}A \quad (1-33a)$$

$$I_2 = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji}) = \frac{1}{2}[(\text{tr}A)^2 - \text{tr}(A^2)] \quad (1-33b)$$

$$I_3 = \xi_{ijk}A_{1i}A_{2j}A_{3k} = \det A \quad (1-33c)$$

It can be proved that  $I_1$ ,  $I_2$ , and  $I_3$  are invariant when the tensor is transformed into its representation into another rotated coordinate system.

Typically, it has three real roots.

$$[A_{ij} - \lambda_{(q)}\delta_{ij}]n_i^{(q)} = 0, \quad (q = 1,2,3) \quad (1-34)$$

and

$$n_i^{(q)}n_i^{(q)} = 1, \quad (q = 1,2,3)$$

Replacing  $A$  with the stress tensor  $\sigma$  or strain tensor  $\varepsilon$  in the above operations, one can find the principal stresses or principal strains. Due to the invariant nature of the three invariants, they are often used in constitutive laws to avoid the violation of objectivity principle.

### 1.5.6 Typical Operators

Operator  $\nabla$  (del) of vector calculus can be represented as:

$$\nabla = \frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3 = \frac{\partial}{\partial x_i}e_i \quad (1-35)$$

If  $\phi$  is a scalar function then  $\nabla\phi = \text{grad}\phi$  is the gradient of the function. It has applications in the plasticity theory for representing the flow rule. The gradient of a scalar function is a vector.

The gradient of a vector field  $v$  is represented as:

$$\nabla v = \frac{\partial v_j}{\partial x_i}e_j \otimes e_i \quad (1-36)$$

It is a tensor.

The divergence of a vector field  $v$  is also scalar. It can be represented as:

$$\nabla \bullet v = \frac{\partial v_j}{\partial x_i}e_i \bullet e_j = \frac{\partial v_j}{\partial x_i}\delta_{ij} = \frac{\partial v_i}{\partial x_i} \quad (1-37)$$

The curl of a vector is a vector and is defined as:

$$\nabla \times v = \xi_{ijk} \frac{\partial v_j}{\partial x_i}e_k \quad (1-38)$$

### 1.5.7 Divergence Theorem

The Integral Theorems of Gauss and Stokes (or the divergence theorem) is widely used in continuum mechanics. It presents the relationship between the volume integral and the surface integral of any quantity. For a tensorial quantity  $T_{ij\dots k}$  and its gradient

$$T_{ij\dots k,q} = \frac{\partial T_{ij\dots k}}{\partial x_q}$$

The theorem states:

$$\int_S T_{ij\dots k} n_q dS = \int_V T_{ij\dots k,q} dV \quad (1-39)$$

For a scalar field  $\lambda$ , it ends with:

$$\int_S \lambda n_q dS = \int_V \lambda_{,q} dV \quad (1-40)$$

For a vector field  $v$ :

$$\int_S v \bullet n dS = \int_V \text{div} v dV \quad \text{or} \quad \int_S v_q n_q dS = \int_V v_{q,q} dV \quad (1-41)$$

$$\int_S n \times v dS = \int_V \text{curl} v dV \quad \text{or} \quad \int_S \xi_{ijk} n_i v_j e_k dS = \int_V \xi_{ijk} \frac{\partial v_j}{\partial x_i} e_k dV \quad (1-42)$$

In 2D spaces, it reduces to the area integral and curve (line) integral.

## 1.6 Fundamentals of Continuum Mechanics

### 1.6.1 Concept of a Continuum and Representative Element

With unaided eyes, metals, glasses, other solids, and liquids look like a continuum; there is no space in them. One can imagine that the gases are filled with molecules everywhere and can also be considered a continuum. The physical concept of continuum is very complicated. There are always voids and minute cracking in solids, and voids in liquids. There are also spaces between atoms and molecules. Even within the atom there are tremendous spaces. However, there might be force fields in these spaces which bond the molecules and atoms. The concept of a continuum in continuum mechanics is both concrete and abstract. In the abstract sense, when the effects due to any discontinuity are negligible, the material may behave like a continuum. Mathematically, if the displacement field of a material can be represented as a continuous function, the material can be considered a continuum. While there is no restriction on the homogeneity assumption of the material, this assumption is usually implied as typically one addresses the material with the same set of constitutive equations. AC is, by nature, a heterogeneous material comprised of asphalt binder, aggregates, and voids. Each of the constituents can be considered as homogeneous. Using a large representative volume element (RVE), the thus obtained properties of RVE can be used to represent the properties of the material at any sizes, including the infinitesimal elements.

## 1.6.2 Description of Kinematics-Deformation Gradient and Finite Strain Tensor

### 1.6.2.1 Coordinate Representation

Figure 1.3 presents the reference configuration and the current configuration of a deformable body. A segment in the reference configuration is represented as:

$$dX = dX_A I_A \quad (1-43)$$

Therefore:

$$(dX)^2 = dX \bullet dX = dX_A dX_A \quad (1-44)$$

Where  $dX_A$  are the three components in the three orthogonal directions represented by  $I_A$  for the reference configuration.

In the current configuration the segment is represented as:

$$dx = dx_i e_i \quad (1-45)$$

Therefore:

$$(dx)^2 = dx \bullet dx = dx_i dx_i \quad (1-46)$$

Where  $dx_i$  are the three components in the three orthogonal directions represented by  $e_i$  for the current configuration.

For a continuum, one can assume a continuous and one-to-one mapping as:

$$x_i = \chi_i(X) \quad (1-47)$$

Therefore:

$$dx_i = \frac{\partial \chi_i}{\partial X_A} dX_A = \chi_{i,A} dX_A \quad (1-48)$$

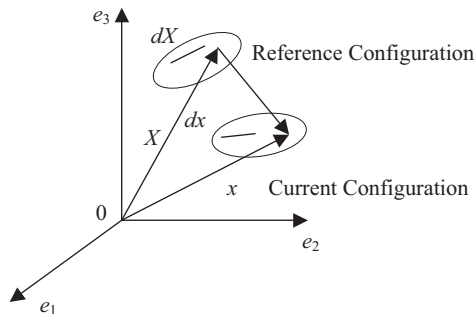
$$\chi_{i,A} \equiv F_{iA} \quad (1-49)$$

is the deformation gradient and

$$dx = F \bullet dX \quad (1-50)$$

$$dX = F^{-1} \bullet dx \quad (1-51)$$

**FIGURE 1.3** Illustration of deformation kinematics.



Considering:

$$\begin{aligned} (dx)^2 - (dX)^2 &= dx_i dx_i - dX_A dX_A = (x_{i,A} dX_A)(x_{i,B} dX_B) - \delta_{AB} dX_A dX_B = (x_{i,A} x_{i,B} - \delta_{AB}) dX_A dX_B \\ &= (C_{AB} - \delta_{AB}) dX_A dX_B \end{aligned} \quad (1-52)$$

Where  $C_{AB} = x_{i,A} x_{i,B}$  or  $C = F^T \bullet F$  in matrix or tensor representation.

The Lagrangian Finite Strain Tensor is defined as:

$$2E_{AB} = C_{AB} - \delta_{AB} \quad (1-53)$$

or

$$2E = C - \delta \quad (1-54)$$

and

$$(dx)^2 - (dX)^2 = (C_{AB} - \delta_{AB}) dX_A dX_B = 2E_{AB} dX_A dX_B \quad (1-55)$$

For the one-to-one mapping, one can represent  $X$  as a function of  $x$ . Therefore:

$$\begin{aligned} (dx)^2 - (dX)^2 &= \delta_{ij} dx_i dx_j - (X_{A,i} dx_i)(X_{A,j} dx_j) = (\delta_{ij} - X_{A,i} X_{A,j}) dx_i dx_j = (\delta_{ij} - c_{ij}) dx_i dx_j \\ c_{ij} &= X_{A,i} X_{A,j} \text{ or } c = (F^{-1})^T \bullet (F^{-1}) \end{aligned} \quad (1-56)$$

The Cauchy Deformation Tensor or Eulerian Finite Strain Tensor is defined as:

$$2e_{ij} = \delta_{ij} - c_{ij} \text{ or } 2e = I - c \quad (1-57)$$

Considering two segments  $dX^{(1)}$  and  $dX^{(2)}$  in the reference configuration,  $dx^{(1)}$  and  $dx^{(2)}$  represent the two segments in the current configuration.

$$\begin{aligned} dx^{(1)} \bullet dx^{(2)} &= F \bullet dX^{(1)} \bullet F \bullet dX^{(2)} = dX^{(1)} \bullet F^T \bullet F \bullet dX^{(2)} \\ &= dX^{(1)} \bullet C \bullet dX^{(2)} = dX^{(1)} \bullet (I + 2E) \bullet dX^{(2)} = dX^{(1)} \bullet dX^{(2)} + dX^{(1)} \bullet 2E \bullet dX^{(2)} \end{aligned} \quad (1-58)$$

If there is no strain, one can have:

$$dx^{(1)} \bullet dx^{(2)} = dX^{(1)} dX^{(2)} \cos \theta = dX^{(1)} \bullet dX^{(2)} = dX^{(1)} dX^{(2)} \cos \theta \quad (1-59)$$

It means the angles between any two segments remain unchanged.

### 1.6.2.2 Displacement Representation

Obviously, the displacement of point  $X$  can be represented as:

$$\begin{aligned} u_i(X_A) &= x_i(X_A) - X_i \\ u_A(x_i) &= x_A - X_A(x_i) \end{aligned} \quad (1-60a, 1-60b)$$

Therefore the Lagrangian Finite Strain Tensor is:

$$2E_{AB} = x_{i,A} x_{i,B} - \delta_{AB} = (u_{i,A} + \delta_{i,A})(u_{i,B} + \delta_{i,B}) - \delta_{AB}$$

or

$$2E_{AB} = u_{A,B} + u_{B,A} + u_{i,A} u_{i,B} \quad (1-61)$$

The Eulerian Finite Strain Tensor is:

$$2e_{ij} = \delta_{ij} - X_{A,i}X_{A,j} = \delta_{ij} - (\delta_{Ai} - u_{A,i})(\delta_{Aj} - u_{A,j})$$

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{A,i}u_{A,j} \quad (1-62)$$

By neglecting the high order terms, one has:

$$2E_{AB} = u_{A,B} + u_{B,A}, \quad 2e_{ij} = u_{i,j} + u_{j,i} \quad (1-63a, 1-63b)$$

Both reduce to the small strain representation.

### 1.6.2.3 Rate of Deformation

The previous sections have focused on the deformation gradients and strains that are based on displacements or locations. They are spatial gradients. This section will discuss time-dependent characteristics of deformation. Instead of using the displacement, the velocity  $v_i$  will be the field variables for the study.

The spatial velocity gradient is defined as:

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \quad (1-64)$$

This tensor can be decomposed into a symmetric part and a skew-symmetric part:

$$L_{ij} = D_{ij} + W_{ij} \quad (1-65)$$

$$D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad W_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \quad (1-66a, 1-66b)$$

The symmetric part  $D_{ij}$  is the rate of deformation tensor and the skew-symmetric part  $W_{ij}$  is the spin tensor representing rigid rotation.

The spatial velocity change can be represented as:

$$dv_i = \frac{\partial v_i}{\partial x_j} dx_j \quad \text{or} \quad dv = L \bullet dx \quad (1-67)$$

Since

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial X_A} \frac{\partial X_A}{\partial x_j} = \frac{d}{dt} \left( \frac{\partial x_i}{\partial X_A} \right) \frac{\partial X_A}{\partial x_j} \quad (1-68)$$

One has:

$$L = \dot{F} \bullet F^{-1} \quad (1-69)$$

or

$$\dot{F} = L \bullet F \quad (1-70)$$

This relationship is often used for deriving other relationships in the following sections.

### 1.6.2.4 Stretch Ratio

The stretch ratio  $\alpha$  is defined as the ratio between the length  $dx$  and that of  $dX$  measured in the reference direction  $N$ . It is a scalar.

$$\alpha = \frac{dx}{dX} \quad (1-71)$$

One may use a definition that can be more conveniently implemented.

$$\begin{aligned} \alpha^2 &= \left(\frac{dx}{dX}\right)^2 \\ (dx)^2 &= dx \bullet dx = F \bullet dX \bullet F \bullet dX = dX \bullet C \bullet dX \\ \alpha^2 &= \left(\frac{dx}{dX}\right)^2 = \frac{dX}{dX} \bullet C \bullet \frac{dX}{dX} = N \bullet C \bullet N \end{aligned} \quad (1-72)$$

One can define the stretch ratio in the  $n$  direction.

$$\frac{1}{\beta} = \frac{dX}{dx} \quad (1-73)$$

$$\begin{aligned} (dX)^2 &= dX \bullet dX = F^{-1} \bullet dx \bullet F^{-1} \bullet dx = dx \bullet c \bullet dx \\ \frac{1}{\beta^2} &= \left(\frac{dX}{dx}\right)^2 = \frac{dX}{dX} \bullet c \bullet \frac{dX}{dX} = n \bullet c \bullet n \end{aligned} \quad (1-74)$$

Due to the deformation,  $n$  may not be a unit vector and therefore typically  $\alpha \neq \beta$ .

### 1.6.2.5 Material-Time Derivative

For any quantity (may be a scalar, vector, or tensor),

$$B_{ij\dots} = B_{ij\dots}(X, t) \text{ or } B_{ij\dots} = B_{ij\dots}(x, t) \quad (1-75a, 1-75b)$$

Its time variation rate can be represented either as:

$$\frac{d}{dt} [B_{ij\dots}(X, t)] = \frac{\partial}{\partial t} [B_{ij\dots}(X, t)] \quad (1-76)$$

or in the spatial reference as:

$$\frac{d}{dt} [B_{ij\dots}(x, t)] = \frac{\partial}{\partial t} [B_{ij\dots}(x, t)] + \frac{\partial}{\partial x_k} [B_{ij\dots}(x, t)] \frac{dx_k}{dt} \quad (1-77)$$

$$\frac{d}{dt} [B_{ij\dots}(x, t)] = \frac{\partial}{\partial t} [B_{ij\dots}(x, t)] + \frac{\partial}{\partial x_k} [B_{ij\dots}(x, t)] v_k \quad (1-78)$$

Therefore, one has the following operator:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k} \quad (1-79)$$

or

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \bullet \nabla_x \quad (1-80)$$

### 1.6.2.6 Changes of Segment, Area, and Volume

#### 1.6.2.6.1 Line Segment Change

$$\dot{dx} = \dot{F} \bullet dX \quad (1-81)$$

$$\dot{F} = L \bullet F \quad (1-82)$$

$$\dot{dx} = L \bullet F \bullet dX = L \bullet dx \quad (1-83)$$

The material derivative of the dot product of  $dx$ :

$$(dx \bullet \dot{dx}) = 2dx \bullet \dot{dx} = dx \bullet 2L \bullet dx = dx \bullet 2(D + W) \bullet dx = dx \bullet 2D \bullet dx \quad (1-84)$$

#### 1.6.2.6.2 Area Change

The area change can be evaluated through an area element composed of two segments  $dX^{(1)}$  and  $dX^{(2)}$ . In the reference configuration the area is:

$$dS_A^o = \xi_{ABC} dX_B^{(1)} dX_C^{(2)} \quad (1-85)$$

In the current configuration:

$$dS_i = \xi_{ijk} dx_j^{(1)} dx_k^{(2)} = \xi_{ijk} x_{j,B} dX_B^{(1)} x_{k,C} dX_C^{(2)}$$

$$x_{i,A} dS_i = \xi_{ijk} x_{i,A} x_{j,B} x_{k,C} dX_B^{(1)} dX_C^{(2)} = \xi_{ijk} F_{iA} F_{jB} F_{kC} dX_B^{(1)} dX_C^{(2)}$$

$$\xi_{ijk} F_{iA} F_{jB} F_{kC} = \xi_{ABC} \det F = \xi_{ABC} J$$

$$x_{i,A} dS_i = \xi_{ABC} J dX_B^{(1)} dX_C^{(2)}$$

Multiply both sides of the above equation by  $X_{A,q}$ , one obtains:

$$x_{i,A} X_{A,q} dS_i = \xi_{ABC} J dX_B^{(1)} dX_C^{(2)} X_{A,q}$$

$$x_{i,A} X_{A,q} = \delta_{iq}$$

$$dS_q = X_{A,q} J dS_A^o \quad (1-86)$$

Considering the identity  $\det B = \text{tr}(\dot{B} \bullet B^{-1}) \det B$  and replace  $B$  with  $F$ , one can obtain:

$$\det \dot{F} = \text{tr}(\dot{F} \bullet F^{-1}) \det F = J \text{tr} L \text{ or } \dot{J} = J \text{div} v \quad (1-87)$$

This relationship is important for defining stress in the reference configuration:

$$dS = J(F^{-1})^T \bullet dS^o$$

$$dS \bullet F = J dS^o \quad (1-88)$$

Therefore:

$$\dot{dS} \bullet F + dS \bullet \dot{F} = J \dot{dS}^0 = J \text{tr} L dS^0 \quad \text{Multiply both sides by } F^{-1}$$

$$\dot{dS} + dS \bullet \dot{F} \bullet F^{-1} = J \text{tr} L dS^0 \bullet F^{-1} = \text{tr} L dS$$

$$L = \dot{F} \bullet F^{-1} \quad \text{One has:}$$

$$\dot{dS} = \text{tr} L dS - dS \bullet L \tag{1-89}$$

**1.6.2.6.3** Volume Change

In the reference configuration:

$$dV^0 = dX^{(1)} \bullet dX^{(2)} \times dX^{(3)} = \xi_{ABC} dX_A^{(1)} dX_B^{(2)} dX_C^{(3)} \tag{1-90}$$

In the current configuration:

$$\begin{aligned} dV &= dx^{(1)} \bullet dx^{(2)} \times dx^{(3)} = \xi_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)} \\ dx^{(1)} &= F \bullet dX^{(1)} \quad dx^{(2)} = F \bullet dX^{(2)} \quad dx^{(3)} = F \bullet dX^{(3)} \\ dx^{(1)} &= x_{i,A} dX_A^{(1)} \quad dx^{(2)} = x_{i,B} dX_B^{(2)} \quad dx^{(3)} = x_{i,C} dX_C^{(3)} \\ dV &= \xi_{ijk} x_{i,A} x_{i,B} x_{i,C} dX_A^{(1)} dX_B^{(2)} dX_C^{(3)} = J dV^0 \\ \dot{dV} &= J \dot{dV}^0 = J(\text{tr} L) dV^0 = (\text{tr} L) dV \end{aligned} \tag{1-91}$$

If the deformation is isochoric, then  $dV = 0 \quad \text{tr} L = 0$

**1.6.2.6.4** Rate of Stretch Ratio

Considering the identity:

$$dx n_i = x_{i,A} dX N_A \tag{1-92}$$

$$n_i \alpha = x_{i,A} N_A$$

or

$$n \alpha = F \bullet N$$

So:

$$\begin{aligned} \dot{n} \alpha + n \dot{\alpha} &= F \bullet N = L \bullet F \bullet N = L \bullet n \alpha \\ \dot{n} + n \dot{\alpha} / \alpha &= L \bullet n \\ n \bullet \dot{n} + n \bullet n \dot{\alpha} / \alpha &= n \bullet L \bullet n \\ (n \bullet n = 1 \quad n \bullet \dot{n} = 0) \\ \dot{\alpha} / \alpha &= n \bullet L \bullet n \end{aligned}$$

OR

$$\begin{aligned}\dot{\alpha} / \alpha &= L_{ij} n_i n_j \\ L_{ij} n_i n_j &= (D_{ij} + W_{ij}) n_i n_j = D_{ij} n_i n_j \\ \dot{\alpha} / \alpha &= n \bullet D \bullet n\end{aligned}\quad (1-93)$$

OR

$$\dot{\alpha} / \alpha = D_{ij} n_i n_j$$

### 1.6.2.7 Rotation Tensor and Stretch Tensor

The deformation gradient tensor  $F$  can be decomposed into a rotational component  $R$  followed by a stretch component  $U$  or stretch component tensor  $V$  followed by a rotational  $R$ . It is named as polar decomposition. Figure 1.4 illustrates this concept.  $PQ$  is deformed into  $pq$  but can be considered as first stretched to  $pq'$  (in parallel to  $PQ$ ) and then rotated to  $pq$ .  $R$  is the orthogonal rotational tensor and  $U$  and  $V$  are symmetric, positive definite right and left stretch tensors.

$dX$  is stretched to  $dx'$  and is then rotated to  $dx$ . Then one can have:

$$dx' = U \bullet dX \quad (1-94)$$

$$dx = R \bullet dx' \quad (1-95)$$

Therefore  $dx = R \bullet U \bullet dX$  and  $F = R \bullet U$ . Similarly, one can obtain  $F = V \bullet R$ .

Therefore:

$$F = R \bullet U = V \bullet R \quad (1-96)$$

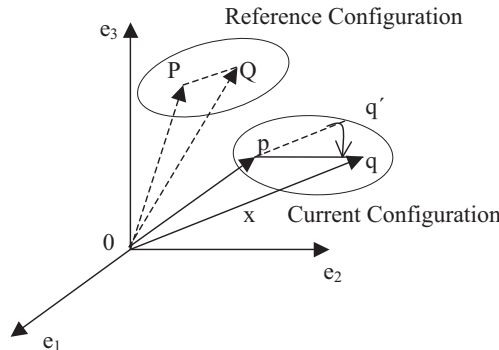
Since  $R$  is purely rotational, one has  $dx \bullet dx = (R \bullet dx') \bullet (R \bullet dx') = dx' R^T R dx' = dx' \bullet dx'$  and  $R^T R = I$ .

Through this relationship and the dot product and strain tensor definitions:

One can prove that:

$$U \bullet U = C \text{ and } F \bullet F^T = V \bullet V \quad (1-97a, 1-97b)$$

**FIGURE 1.4** Illustration of polar decomposition.



### 1.6.3 Stresses

#### 1.6.3.1 Cauchy Stress

Considering an area element  $\Delta S$  of current configuration, having applied forces  $\Delta f_i$  and momentum  $\Delta M_i$  on it, the Cauchy stress principle asserts that the following limits (as the area  $\Delta S$  approaches zero) exist:

$$\lim_{\Delta S \rightarrow 0} \frac{\Delta f_i}{\Delta S} = \frac{df_i}{dS} = t_i^{(n)} \quad (1-98)$$

It is named the stress vector or traction vector.

$$\lim_{\Delta S \rightarrow 0} \frac{\Delta M_i}{\Delta S} = 0 \quad (1-99)$$

This means that the distributed momentum is equal to zero.

#### 1.6.3.2 Stress Tensor

Consider an infinitesimal tetrahedron in Figure 1.5 at point  $P$  in the current configuration, the three stress vectors can be represented as:

$$\begin{aligned} t^{(e_1)} &= t_1^{(e_1)} e_1 + t_2^{(e_1)} e_2 + t_3^{(e_1)} e_3 \\ t^{(e_2)} &= t_1^{(e_2)} e_1 + t_2^{(e_2)} e_2 + t_3^{(e_2)} e_3 \\ t^{(e_3)} &= t_1^{(e_3)} e_1 + t_2^{(e_3)} e_2 + t_3^{(e_3)} e_3 \end{aligned} \quad (1-100)$$

Using summation convention, these stress vectors can be represented as:

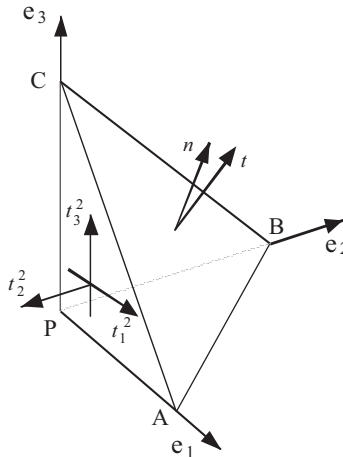
$$t^{(e_i)} = t_j^{(e_i)} e_j, \quad (i=1,2,3)$$

In Figure 1.5, the minus sign denotes the opposite direction of the stress vectors.

The Newton second law for the tetrahedron can be represented as:

$$t_i^{(n)} dS - t_i^{(e_1)} dS_1 - t_i^{(e_2)} dS_2 - t_i^{(e_3)} dS_3 + \rho b_i dV = \rho a_i dV \quad (1-101)$$

**FIGURE 1.5** Free-body diagram for illustrating stress tensor.



Where  $dV$  is the volume of the infinitesimal tetrahedron.

Dividing both sides of the above equation by  $dS$ , and considering  $n_i = \frac{dS_i}{dS}$  and  $\frac{dV}{dS} = 0$  when  $dS$  approaches zero, one obtains:

$$t_i^{(n)} = t_i^{(e_1)}n_1 + t_i^{(e_2)}n_2 + t_i^{(e_3)}n_3 \quad (1-102)$$

Where  $i = 1,2,3$ , therefore the above equation actually represents three equations. The nine stress vector components  $t_i^{(e_j)}$  as a set of quantities represent the Cauchy stress tensors. It is represented as  $\sigma_{ij} = t_i^{(e_j)}$  ( $i = 1,3; j = 1,3$ ). Figure 1.6 is an illustration of the nine components of a stress tensor.

The stress vector on the plane with normal direction  $n$  can then be generalized as:

$$t_i^{(n)} = \sigma_{ji}n_j \quad (1-103)$$

or

$$t^{(n)} = n \bullet \sigma \quad (1-104)$$

### 1.6.3.3 Stress Tensor Transformation in Different Coordinate Systems

For the same stress vector, it can be represented using the components in two different coordinate systems, the  $e_i$  and  $e_i'$  systems. Logically (it is the same vector but different representations), the following relationship will hold.

$$t^{(n)} = t_i^{(n)}e_i = t_r^{(n')}e_r'$$

Considering  $t_i^{(n)} = \sigma_{ji}n_j$  and  $t_r^{(n')} = \sigma'_{rs}n'_s$

One has the following:

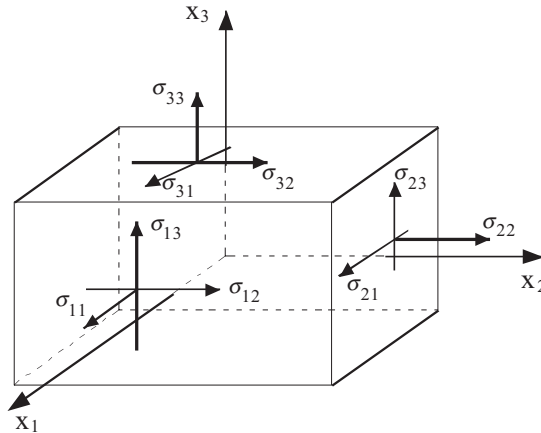
$$t^{(n)} = \sigma_{ji}n_j e_i = \sigma'_{rs}n'_r e'_s$$

$$n'_r = T_{rl}n_l \text{ and } e'_s = T_{sk}e_k$$

After some manipulations of the summed indices:

$$\sigma_{ij}n_j e_i = T_{rl}T_{sk}\sigma'_{rs}n_l e_k$$

**FIGURE 1.6** Schematic diagram for components of stress tensor.



Please note that  $n_l e_k$  represents the nine independent bases and can be replaced as  $n_j e_i$ . By replacing  $l$  with  $j$  and  $k$  with  $i$ , one has:

$$\begin{aligned}\sigma_{ij} n_j e_i &= T_{rj} T_{si} \sigma'_{rs} n_j e_i \\ \sigma'_{rs} &= T_{sr} T_{rj} \sigma_{ij}\end{aligned}\quad (1-105)$$

In matrix format, it can be represented as:

$$\sigma' = T \sigma T^T \quad (1-105a)$$

#### 1.6.3.4 Stress Invariants

Following Section 2.5, the three invariants of the stress tensor are:

$$I_1 = \sigma_{ii} = \text{tr} \sigma \quad (1-106)$$

$$I_2 = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ji}) = \frac{1}{2} [(tr \sigma)^2 - tr(\sigma^2)] \quad (1-107)$$

$$I_3 = \xi_{ijk} \sigma_{1i} \sigma_{2j} \sigma_{3k} = \det \sigma \quad (1-108)$$

#### 1.6.3.5 Symmetry

Due to the non-existent distributed momentum (the second principle of Cauchy Stress), the stress tensor thus defined will be symmetric. Section 3.4.5 will present a proof.

#### 1.6.3.6 Principal Stresses

For symmetric real valued tensors, there exist three directions where the stress vectors are coincident with the direction of the normal of the planes. These three stresses are the principal stresses. One can use Equations 1-31, 1-32, and 1-34 to find the principal directions and principal stresses.

#### 1.6.3.7 Maximum and Minimum Stresses

As the stress vector is  $t^{(n)} = n \bullet \sigma$ , its component along the normal of the surface is:

$$\sigma^N = t^{(n)} \bullet n = n \bullet \sigma \bullet n = \sigma_{ij} n_i n_j \quad (1-109)$$

Note that  $\sigma_N$  is the magnitude of the stress vector projected on the normal direction. The magnitude of the stress vector projected on the tangent direction is:

$$\sigma_T^2 = t^{(n)} \bullet t^{(n)} - \sigma_N^2 \quad (1-110)$$

The maximum or minimum are applicable to  $\sigma_N$  or  $\sigma_T$ . Taking  $\sigma_N$  as an example, to find maximum or minimum  $\sigma_N$  is equivalent to find the maximum or minimum of the function:

$$f(n_i) = \sigma_{ij} n_i n_j \quad (1-111)$$

The Lagrangian Multiplier method can be used to find the solution. One can constitute an equation  $g(n_i) = \sigma_{ij} n_i n_j - \zeta(n_i n_i - 1)$  with  $n_i n_i - 1 = 0$ . With the necessary condition of the extremal conditions, one will have:

$$\frac{\partial g}{\partial n_k} = \sigma_{ij} \delta_{ik} n_j + \sigma_{ij} \delta_{jk} n_i - 2\zeta \delta_{ik} n_i = 0 \quad (1-112)$$

Considering the symmetry of the stress tensor and the replacing function of the Delta operator, one can have:

$$\sigma_{ij} \delta_{ik} n_j = \sigma_{kj} n_j = \sigma_{ij} \delta_{jk} n_i = \sigma_{ik} n_i$$

Therefore:

$$(\sigma_{ik} - \zeta \delta_{ik}) n_i = 0 \quad (1-113)$$

Comparing it with Equation 1-34, it is the same equation. Therefore, the principal stresses are actually the maximum and minimum stresses. Following the similar procedure, one can determine the maximum shear stresses.

### 1.6.3.8 Deviatoric and Hydraulic Stresses

The mean of the sum of the three normal stresses ( $I_1 = \sigma_{ii} = tr \sigma$ ) is the hydraulic stress or spherical stress. It can be represented as an isotropic stress tensor:

$$\sigma_{ij}^M = \frac{1}{3} I \delta_{ij} \quad (1-114)$$

The difference between the stress tensor and its mean stress tensor is called the deviatoric stress tensor.

$$S_{ij} = \sigma_{ij} - \sigma_{ij}^M \quad (1-115)$$

### 1.6.3.9 Octohedral Stress

The plane whose normal makes equal angle to the directions of the principal stresses is called the octahedral plane. The directional cosines of this plane are equal to  $1/\sqrt{3}$  and therefore  $\sigma^N = \frac{1}{3} \sigma_{ii}$

$$\sigma_T = \sigma_{oct} = \frac{1}{3} \sqrt{[\sigma_1 - \sigma_2]^2 + [\sigma_2 - \sigma_3]^2 + [\sigma_3 - \sigma_1]^2} \quad (1-116)$$

Where  $\sigma_1$   $\sigma_2$   $\sigma_3$  are the three principal stresses.

### 1.6.3.10 Piola-Kirchhoff Stresses

In calculating the engineering stresses, one often uses the force applied on a surface normalized with the surface area in the reference configuration (the original surface). The first Piola- Kirchhoff (P-K) stress is such a measure.

$$\lim_{\Delta S^o \rightarrow 0} \frac{\Delta f_i}{\Delta S^o} = \frac{df_i}{dS^o} = p_i^N \quad (1-117)$$

It should be noted that the direction  $N$  is also a measure in the reference frame. The first P-K stress is defined in such a way that:

$$\begin{aligned} p_i^N &= P_{Ai}^{1st} N_A \\ df_i &= \sigma_{ji} n_j dS \\ df_i &= P_{Ai}^{1st} N_A dS^o \end{aligned} \quad (1-118)$$

Considering the area transform identity:

$$\begin{aligned} n_i dS &= X_{A,i} J N_A dS^o \\ df_i &= \sigma_{ji} n_j dS = \sigma_{ji} X_{A,j} J N_A dS^o \\ P_{Ai}^{1st} &= \sigma_{ji} X_{A,j} J \end{aligned}$$

Since

$$X_{A,j} = F^{-1}$$

One has

$$P_{Ai}^{1st} = \sigma_{ji} F_{Aj}^{-1} J \quad (1-119)$$

The second Piola-Kirchhoff stress tensor refers a measure of the forces in the reference frame to the area in the reference frame.

$$df_B^o = P_{AB}^{2nd} N_A dS^o$$

Considering the unit normal  $n$  versus  $N$  as vector element in the domain, one would have:

$$\begin{aligned} df_i &= F_{iB} \bullet df_B \\ df_i &= \sigma_{ji} n_j dS = \sigma_{ji} X_{A,j} J N_A dS^o = F_{iB} \bullet df_B = F_{iB} \bullet P_{AB}^{2nd} N_A dS^o \end{aligned}$$

Therefore:

$$P_{AB}^{2nd} = F_{iB}^{-1} \sigma_{ij} F_{Aj}^{-1} J \quad (1-120)$$

## 1.6.4 Fundamental Continuum Mechanics Equations

### 1.6.4.1 General Concepts

There is a set of equations that a continuum must observe during its deformation process. These equations include the conservation equations involving mass, momentum, angular momentum, energy; the second Law of Thermodynamics; free energies; the objectivity assumptions; the strain compatibility conditions; and constitutive laws.

The divergence theorem is widely used in deriving the differential format of the conservation laws. It is represented in the following three formats in terms of a scalar field, a vector field, and a tensor field.

$$\iiint_V \eta_i dV = \iint_{\partial V} \eta n_i dS \quad \iiint_V \frac{\partial a_i}{\partial x_i} dV = \iint_{\partial V} a_i n_i dS \quad \iiint_V \frac{\partial A_{ij}}{\partial x_i} dV = \iint_{\partial V} A_{ij} n_i dS \quad (1-121a, b, c)$$

### 1.6.4.2 Density Definition

Considering a volume element  $\Delta V$  surrounding point  $P$ , the amount of mass contained in  $\Delta V$  is  $\Delta m$ , the average density of this volume element is:

$$\rho_{ave} = \frac{\Delta m}{\Delta V} \quad (1-122)$$

When  $\Delta V$  approaches infinitesimal, it becomes the density  $\rho$  at point  $P$ . In reality, the infinitesimal may not be meaningful when the volume element is smaller than a few molecules or atoms.

The density is a scalar function of position and time and thus may vary from point to point within a given body.

$$\rho = \rho(x_i, t)$$

or

$$\rho = \rho(x, t)$$

#### 1.6.4.3 Mass Conservation (Material Derivative)

For any domain, the mass cannot be destroyed or produced. Therefore, the mass flowing in or out of a specific volume and its local mass change (within the volume) will sum up to zero.

$$\iint_S \rho v \cdot n dS + \frac{\partial}{\partial t} \iiint_V \rho dV = 0 \quad (1-123)$$

Considering that:

$$\iint_S \rho v \cdot n dS = \iiint_V \frac{\partial(\rho v_i)}{\partial x_i} dV$$

If the above equations hold for any domain, one has the following equation:

$$\begin{aligned} \frac{\partial(\rho v_i)}{\partial x_i} + \frac{\partial}{\partial t} \rho &= 0 \\ \frac{\partial(\rho v_i)}{\partial x_i} &= \frac{\partial \rho}{\partial x_i} v_i + \rho \frac{\partial v_i}{\partial x_i} \end{aligned}$$

Considering the material derivative in Section 3.2.5, one has:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} v_i$$

Therefore, the mass conservation or the continuity equation can be represented as:

$$\frac{d\rho}{dt} + \rho \frac{\partial v_i}{\partial x_i} = 0 \quad (1-124)$$

#### 1.6.4.4 Linear Momentum Equilibrium Equation

For any finite domain (body), the force equilibrium equation can be represented as:

$$\iint_S t^{(n)} dS + \iiint_V \rho b dV = 0 \quad (1-125)$$

Its indicial format:

$$\iint_S n_i \sigma_{ij} dS + \iiint_V \rho b_j dV = 0$$

By applying the divergence theorem, one has:

$$\iiint_V \left( \frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j \right) dV = 0$$

If the above equation holds for any of the domain, the following will also hold:

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j = 0 \quad (1-126)$$

### 1.6.4.5 Angular Momentum

For any finite domain (body), the angular momentum equilibrium equation can be represented as:

$$\iint_S x \times t^{(n)} dS + \iiint_V \rho x \times b dV = 0 \quad (1-127)$$

The indicial format can be represented as:

$$x \times t = e_{pqr} x_p t_q e_r = e_{pqr} x_p n_j \sigma_{jq} e_r$$

$$x \times b = e_{pqr} x_p b_q e_r$$

Using the divergence theorem for surface integral one obtains:

$$\iiint_V e_{pqr} \left\{ \frac{\partial}{\partial x_j} (x_p \sigma_{jq}) + \rho x_p b_q \right\} dV = 0$$

$$e_{pqr} \left\{ \frac{\partial (x_p \sigma_{jq})}{\partial x_j} + \rho x_p b_q \right\} = 0$$

$$e_{pqr} \left\{ x_p \left( \frac{\partial \sigma_{jq}}{\partial x_j} + \rho b_q \right) + \sigma_{pq} \right\} = 0$$

Since:

$$\frac{\partial \sigma_{jq}}{\partial x_j} + \rho b_q = 0$$

$$e_{pqr} \sigma_{pq} = 0$$

or

$$\sigma_{pq} = \sigma_{qp} \quad (1-128)$$

This actually means that stress tensor is symmetric. The condition is that there is no distributed angular momentum.

### 1.6.4.6 Energy Conservation

The energy conservation principle can be stated as: for any domain of interest, the rate of change of the kinetic and internal energy is equal to the rate of work done by the surface tractions and body forces, plus other rates of energies entering or leaving the surface. Other energies may include all types of energies but are limited to thermal and mechanical energies in this book. While a small section may focus on some polar medium, the majority of the book deals with non-polar media.

$$\text{The kinetic energy in any domain: } K(t) = \frac{1}{2} \int_V \rho v_i v_i dV \quad (1-129)$$

$$\text{The rate of work by surface traction: } \int_{\partial V} t_i v_i dS \quad (1-130)$$

The rate of work by body force:  $\int_V \rho b_i v_i dV$  (1-131)

The power of mechanical force is:  $P(t) = \int_V \rho b_i v_i dV + \int_{\partial V} t_i v_i dS$  (1-132)

The rate of thermal energy includes heat supply  $r$  (heat supply per unit mass per unit time) and heat flux  $q$  (heat flux per unit surface area per unit time). It can be represented as:

$$Q(t) = \int_V \rho r dV - \int_{\partial V} q_i n_i dS \quad (1-133)$$

The rate of internal energy  $U$  accounts for all the other energies stored in the medium. It is abstract at this stage. The specific internal energy  $u$  can be introduced so that:

$$U(t) = \int_V \rho u dV \quad \dot{U}(t) = \int_V \rho \dot{u} dV \quad (1-134a, b)$$

Using the above terms, the energy conservation law can be represented as:

$$\dot{U}(t) + \dot{K}(t) = P(t) + Q(t) \quad (1-135)$$

$$\dot{K}(t) = \frac{d}{dt} \int_V \frac{1}{2} \rho v_i v_i dV = \int_V \rho v_i \dot{v}_i dV$$

By using the divergence theorem and the stress-surface traction relationship, one can convert the surface integral  $\int_{\partial V} t_i v_i dS$  into a volume integral:

$$\int_{\partial V} t_i v_i dS = \int_{\partial V} \sigma_{ij} n_j v_i dS = \int_V \frac{\partial(\sigma_{ij} v_i)}{\partial x_j} dV = \int_V \left( \frac{\partial \sigma_{ij}}{\partial x_j} v_i + \sigma_{ij} \frac{\partial v_i}{\partial x_j} \right) dV$$

$$\int_{\partial V} q_i n_i dS = \int_V \frac{\partial q_i}{\partial x_i} dV = \int_V \text{div}(q) dV$$

$$\int_V \rho \dot{u} dV + \int_V \rho v_i \dot{v}_i dV = \int_V \rho b_i v_i dV + \int_V \left( \frac{\partial \sigma_{ij}}{\partial x_j} v_i + \sigma_{ij} \frac{\partial v_i}{\partial x_j} \right) dV + \int_V \rho r dV - \int_V \text{div}(q) dV$$

$$\int_V \rho \dot{u} dV = \int_V \left( -\rho \dot{v}_i + \rho b_i + \frac{\partial \sigma_{ij}}{\partial x_j} \right) v_i dV + \int_V \sigma_{ij} \frac{\partial v_i}{\partial x_j} dV + \int_V \rho r dV - \int_V \text{div}(q) dV$$

By using the momentum equation,  $\left( -\rho \dot{v}_i + \rho b_i + \frac{\partial \sigma_{ij}}{\partial x_j} \right) = 0$ , and

$$\int_V \sigma_{ij} \frac{\partial v_i}{\partial x_j} dV = \int_V \sigma_{ij} (D_{ij} + W_{ij}) dV = \int_V \sigma_{ij} D_{ij} dV$$

The energy equation will reduce to:

$$\rho \dot{u} - \sigma : D - \rho r + \text{div}(q) = 0 \quad (1-136)$$

**1.6.4.7 Objectivity**

The objectivity of constitutive modeling includes the Principle of Equipresence, which states that an independent variable assumed to be present in one constitutive equation of a material should be assumed to be present in all constitutive equations of the same material, unless its presence contradicts an assumed symmetry of the material, or contradicts the principle of material frame-indifference or some other fundamental principle.

There are three other fundamental postulates including the Principle of Determinism for stress, the Principle of Local Action, and the Principle of Material Frame-Indifference (Malvern, 1969). The Principle of Material Frame-Indifference is described as follows.

The material frame-indifference principle actually states that an event  $\{x,t\}$  (location and time) in frame  $x$  should be observed the same by the observers in another frame  $x^*$  as (vector transformation law):

$$\left. \begin{aligned} x^* &= c(t) + Q(t) \bullet x \\ t^* &= t - a \end{aligned} \right\} \tag{1-137}$$

Vectors  $v$

$$v^* = Q(t) \bullet v \tag{1-138}$$

Second-order tensors  $T$  or  $S$  regarded as linear vector transformations:

$$u = T \bullet v \text{ or } u = v \bullet S$$

$$\left. \begin{aligned} T^* &= Q(t) \bullet T \bullet Q(t)^T \\ S^* &= Q(t) \bullet S \bullet Q(t)^T \end{aligned} \right\} \tag{1-139}$$

Deformation gradient  $F(X,t)$

$$F^* = Q(t) \bullet F \tag{1-140}$$

(This two-point tensor transforms like a vector under change of frame at time  $t$ )

Motion of a medium:

$$\begin{aligned} x &= \chi(X,t) \\ x^* &= \chi^*(X,t^*) = c(t) + Q(t) \bullet \chi(X,t), t^* = t - a \\ x^* &= c + Q \bullet x \end{aligned} \tag{1-141}$$

Velocity:

$$v^* = \frac{dx^*}{dt} = \frac{dc}{dt} + Q \bullet x + Q \bullet \frac{dx}{dt} \tag{1-142}$$

### 1.6.5 Green Function

If  $L$  is a linear operator, a partial (or ordinary) differential equation may be written as:

$$L u(x,y,z) = f(x,y,z)$$

As  $L$  is a linear operator, it satisfies the following conditions for linear operator:

For two functions  $f_1(x,y,z)$  and  $f_2(x,y,z)$ , and their corresponding solutions  $u_1(x,y,z)$  and  $u_2(x,y,z)$ , and two scalars  $\alpha_1$  and  $\alpha_2$ , the solution to  $L u(x,y,z) = \alpha_1 f_1(x,y,z) + \alpha_2 f_2(x,y,z)$  will be  $\alpha_1 u_1(x,y,z) + \alpha_2 u_2(x,y,z)$ . This is an important feature. If it is known the solution  $u^*$  to a unit source at point  $(x_0, y_0, z_0)$ , that is  $f(x,y,z) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$ , then one can integrate the solution  $u^*$  to obtain the solution to any source  $z(x_0, y_0, z_0)$ .

Green functions are widely used in the Boundary Element Method. Interested readers may find more detailed description in Qin (2007).

## Recommended Books for Further Reading

The organization of the fundamental theory part of this book was very much inspired by the following books. If readers need more detailed descriptions of the fundamental theories, these books will be very helpful.

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Zienkiewicz, O.C. (1977). *The Finite Element Method*. 3<sup>rd</sup> Edition. McGraw-Hill UK Limited, New York.

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## Suggested Readings

This chapter is mainly based on the teaching notes of the author. These notes are prepared following the presentations from two excellent textbooks by Malvern (1969) and Mase and Mase (1991). Malvern (1969) gave excellent descriptions about the history of the development of continuum mechanics and the physics of the concepts. The book by Mase and Mase presented the major concepts in continuum mechanics in a concise and mathematic manner. The problems for excises are also excellent. If readers need more backgrounds, please read these two books. For convenience and connections, the symbols adopted in this chapter are consistent with those used in these books.

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